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MATHEMATICAL METHODS IN THE STUDY OF WAVE PROPAGATION IN INHOMOGENEOUS MEDIA

By

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TECHNICAL REPORT NO. 1

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Rome Air Development Center
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MICROWAVE PHYSICS LABORATORY

GENERAL TELEPHONE AND ELECTRONICS LABORATORIES, INC.

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ABSTRACT

A study has been made of methods of solution of Maxwell's equations in inhomogeneous isotropic plasmas. Results of that study are given here. The results include a discussion of a general method of reduction, for a variety of coordinate systems, of Maxwell's equations to scalar differential equations for the separate field components. In cartesian coordinates, reflection and transmission coefficients are obtained for plane waves propagating obliquely through a stratified slab of plasma of finite thickness and through a semi-infinite stratified plasma. Some discussion is included concerning total reflection and Brewster's Law in connection with stratified media.

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MATHEMATICAL METHODS IN THE STUDY OF WAVE PROPAGATION IN INHOMOGENEOUS MEDIA

1. INTRODUCTION.

The material to be discussed in this report has been developed to deal with some mathematical aspects of the interaction of electromagnetic waves with ionized media. It is concerned primarily with methods of analysis of Maxwell's Equations in a material medium, together with such additional equations as may be needed to characterize the interactions of the waves with the media, with the aim of obtaining from the set of equations certain formulae describing the way in which the waves are reflected from, absorbed in, or transmitted through the medium. In addition, questions of energy propagation and wave front analysis are considered. The emphasis throughout is on mathematical methods for describing the behavior of waves in media whose electromagnetic properties vary in space. The methods, though originally developed for the analysis of electromagnetic wave propagation in inhomogeneous plasmas, may equally well be applied to inhomogeneous dielectrics or to the study of acoustical wave propagation in inhomogeneous fluids.

The system of equations which will be examined in this report will be assumed linear. The latter assumption is a drastic one. However, inasmuch as the solution of systems of nonlinear equations, even in the case of homogeneous media, involves the use of analytical tools poorly understood even by specialists in the field of nonlinear phenomena, it seems wise to tackle first the problems associated with linear equations in inhomogeneous media. The latter sometimes present knotty problems with which we hope to deal in this report, but the analytical tools (e.g., WKB methods) available for their treatment are much better understood.

The introduction of Maxwell's Equations into the analysis, and their subsequent manipulation, leads to a vector differential equation for the electric field E.* The latter equation is equivalent to a set of three scalar equations, with variable coefficients, in which the components of E are "coupled." The "uncoupling" of these field components, that is, the derivation of scalar differential equations for the individual field components, is, in general, difficult, especially when one is dealing with coordinate systems other than cartesian. In Appendix III we have given a procedure for "uncoupling" the field components. The procedure is certainly not applicable to the general case but assumed that the medium in question varies as a function of only one space coordinate. Also, the orthogonal curvilinear coordinate system to which the procedure applies is not the most general. However, in spite of these limitations, the procedure seems to be of sufficiently wide application to merit consideration.

In the treatment of reflection and transmission coefficients a difficulty arises in the case of spatially non-constant media in that the prescribing of the solutions of the relevant differential equations may require, when one is discussing a semi-infinite medium, the prescribing of boundary conditions at infinity, conditions which are not always easily justifiable. A method of approach to this difficulty of prescribing boundary conditions at infinity is presented in Appendix V. While this

* Capital letters without subscripts will be used to represent vectors in this report. Otherwise, all symbols refer to scalars.

method also presents difficulties of interpretation, it appears to shed some light on the source of the boundary-condition difficulty just mentioned and it also appears to be superior to any method presented in the literature on wave-propagation analysis in inhomogeneous media.

II. DISCUSSION OF THE FUNDAMENTAL EQUATIONS.

In the subsequent discussion, MKS units will be used throughout. The behavior of the electromagnetic disturbance in the medium under consideration will be described mathematically by Maxwell's Equations. The ionized medium under consideration will be considered to be a dilute plasma at low pressures and isotropic, so that, among other things, polarizability effects can be neglected, and the dielectric permittivity and magnetic permeability of the medium will have, to a good approxmation, the values which those quantities have in the vacuum, and these values will be denoted by ϵ_o and μ_o, respectively. Then it will be possible to write down the so-called "constitutive relations" $D = \epsilon_o E$ and $B = \mu_o H$. Hence, the only way in which the influence of the material medium is introduced into the problem is through the current density J appearing in one of Maxwell's Equations. The introduction of the constitutive relations into Maxwell's Equations leads to the pair of equations

$$\nabla \times H = \epsilon_o \frac{\partial E}{\partial t} + J \qquad \text{(a)}$$

$$\nabla \times E = -\mu_o \frac{\partial H}{\partial t}. \qquad \text{(b)} \qquad (1)$$

The field variable H can be eliminated from the latter pair of equations by the familiar procedure of taking the curl of the second equation and replacing the resulting quantity $\nabla \times H$ by its equivalent from the first equation. The equation resulting from the elimination of H is

$$\nabla \times \nabla \times E = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \mu_o \frac{\partial J}{\partial t}. \tag{2}$$

The interesting solutions of (2) are usually the so-called "steady-state" solutions in which the field quantities are regarded as varying in time with a single frequency. Hence, it will be assumed that, for any field variable F, the time variation of F will be described by the relation

$$\frac{\partial F}{\partial t} = - i\omega F, \tag{3}$$

where the radian frequency ω is assumed to be independent of both time and space. With the assumption given in (3), Equation (2) becomes

$$\nabla \times \nabla \times E = \frac{\omega^2}{c^2} E + i\omega\mu_o J. \tag{4}$$

The derivation of J as a function of E, together with some discussion of the assumptions used in the derivation, is presented in Appendix I. The relationship finally obtained for substitution into (4) is

$$J = \frac{i\omega_p{}^2\epsilon_o}{\omega + i\nu} E. \tag{5}$$

Substitution of (5) into (4), followed by some re-ordering of the terms, yields

$$\nabla \times \nabla \times E = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p{}^2}{\omega(\omega + i\nu)}\right] E. \tag{6}$$

Since a good deal of the analysis to follow will not depend specifically on the form of the coefficient of E in Equation (6), it will be convenient to let

$$K = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^{\,2}}{\omega(\omega + i\nu)}\right] \tag{7}$$

and Equation (6) then becomes

$$\nabla \times \nabla \times E = kE. \tag{8}$$

The coefficient k in Equation (8) is now, in general to be considered a complex function of the space coordinates. When k is constant, or nearly constant the solution to (8) can be discussed in terms of the "geometrical optical" approximation in which wave fronts and rays can be described through the use of Hamilton's characteristic function S (eikonal of Bruns), discussed in Appendix II. The quantity k plays the role of a spatially varying index of refraction. However, as k gets close to zero the geometrical optical approximation becomes less and less accurate, and some new method of approximation must be used. In general, however, the problem of ascertaining the behavior of an electromagnetic wave in an inhomogeneous medium requires a detailed solution of the vector wave equation (8), the features of the solution depending strongly on the prescribed functional form of k. A more detailed discussion of the implications of various approximate solutions can be found in reference 2.

III. PRELIMINARY DISCUSSION OF THE PROCEDURE IN SOLVING THE FUNDAMENTAL EQUATION.

As yet, no commitment has been made in the above discussion to the following, which will be discussed in the order given:

A. The restrictions needed on the functional form of k.

B. The types of waves (plane, cylindrical, spherical, etc.) in terms of which the solutions to Equation (8) may be constructed.

C. The shapes of the boundaries of the media with which the waves interact. Media may be assumed infinite in one or more directions.

D. The choice of coordinate systems in terms of which the solutions to (8) can be expressed.

E. The boundary conditions which must be satisfied by the solutions to (8).

Each of the preceding five topics will now be discussed in turn.

A. The Functional Form of k.

Equation (8) is a vector equation for the unknown electric field distribution E. As such, the equation is equivalent to a set of three simultaneous linear scalar partial differential equations for three unknown variables, the mutually orthogonal components of E, relative to a prescribed three-dimensional orthogonal coordinate system. In general, the components of E are said to be "coupled" in each of the three aforementioned scalar equations. In Appendix III, a discussion is given of the system in special cases.

The following observations should be made, however, concerning the system III(c) presented in Appendix III.

1. Solutions of III(c) which are amenable to further manipulation for the mathematical derivation of such "observables" as phase shift, absorption, reflection coefficients, transmission coefficients, etc., generally require that two of the three unknown field components be eliminated from the system III(c), yielding a single scalar linear homogeneous partial differential equation for a single field component.

Alternatively, one may introduce new field variables, usually called "potentials" (Hertz potential, scalar and vector potentials) and obtain scalar equations for each of them.

2. The carrying out of the aforementioned elimination is impossible, in general, when the coordinate system and the functional form of k are arbitrarily prescribed. In fact, the elimination appears, in the present state of the art, to be feasible, in general, only in the case in which k is a function of only one coordinate.

3. If the dependence of k and of E on the coordinates is not suitably chosen to conform to the shapes of the bounding surfaces of the media in question, a further difficulty arises in setting up and solving the relations engendered by the boundary conditions at the bounding surfaces.

For the reasons just given, the only inhomogeneous medium problems which have so far yielded substantially to mathematical and numerical treatments are those involving so-called "stratified media." The latter media may be regarded as being built up of "lamina" of homogeneous material, or they may be continuously varying in one coordinate direction.

4. In discussions of homogeneous media it is frequently desirable to resolve the solutions of (8) into "outgoing" and "incoming" waves. In the very simple situation involving two homogeneous media separated by an infinite plane interface, the incoming wave in one of the media is often discarded completely, the physical reason being that no source of energy exists at infinity in that medium. However, even in the simplest case of waves propagating in the direction of variation of a medium varying in

only one direction, it is not always possible to express the solution of (8) unambiguously in terms of two solutions, one of which can be identified as an "outgoing" disturbance, while the other can be regarded as an "incoming" disturbance. Some examples illustrating this point are cited in Appendix IV.

B. Types of Waves Used in Setting Up Solutions.

The first and most important remark to be made here is that a wave of one type can be represented as a (usually infinite) sum of waves of another type. Specifically, a spherical wave can be represented as the sum of plane waves (cf. ref. 3, p. 238 and ref. 6, chap. VI). This amounts to writing the expression for the spherical wave as a Fourier series or as a Fourier integral. Similar remarks can be made for other types of waves. The selection of the type of wave in terms of which the solution to the wave equation will be expressed will generally be determined by shapes of the boundaries at which boundary conditions are prescribed. Thus cylindrical boundaries will, in general, lead to the choice of cylindrical waves as the most convenient "basis" for the solution.

C. Boundary Shapes.

The vast majority of boundaries encountered in wave problems are either plane, cylindrical, or spherical. Many problems of practical interest can be framed, perhaps as an approximation, in such a way that the boundaries in question are plane. Furthermore, while problems involving other boundary shapes can be solved, their solution involves the use of more sophisticated analysis (notably intricate contour integrations in the complex plane) which would tend to bury the physical conclusions in a welter of analytical detail. Therefore, the surfaces at which boundary conditions will be prescribed in this report will always be taken to be plane.

D. Choice of Coordinate System.

It is a familiar fact that the solution to boundary value problems is often facilitated when the coordinate system is chosen so that the boundary surfaces are coordinate surfaces. Since this report is committed only to the discussion of boundary value problems involving plane boundaries, all boundary value problems in the report will be treated in terms of cartesian coordinates. However, some attention will be given, especially in Appendix III, to the manipulation of Equation (8) in other coordinate systems.

E. Boundary Conditions.

Following the conclusions arrived at in Stratton (ref. 6, p. 37), we shall assume that the tangential components of E and H are continuous across any boundary at which the conductivity is finite. The continuity conditions will be used in Appendix V, where reflection and transmission coefficients are derived for plane waves. More difficult decisions are posed in considering the behavior of the solutions to the wave equation at infinity in cases in which the inhomogeneous medium extends to infinity in the direction of the gradient of k. In view of the conclusions implied by the discussion given in Appendix IV, only those cases are considered in which the solutions of the wave equation have, for large values of the argument, asymptotic forms which have a wave-like character.

IV. FURTHER DISCUSSION OF THE SOLUTION OF THE FUNDAMENTAL EQUATION. REDUCTION TO EQUATIONS FOR EACH FIELD COMPONENT.

In Appendix III a scalar partial differential equation has been derived, under certain restrictive assumptions, for each of the components of the electric field E. The results are summarized below. For each coordinate system the scale factors (described in Appendix III) are given, and the assumptions are given under which the differential equations for the various field components were derived. The Differential equations are special cases of III(j), III(k), and III(p).

A. Cartesian Coordinates (x,y,z). Scale factors: $h_x = h_y = h_z = 1$.

The quantity k is assumed to be a function of x only. The field components E_x, E_y, E_z are assumed to be independent of y. They satisfy the following equations:

$$(D_x^2 + D_z^2 + k)\, E_y = 0 \qquad \text{(a)}$$

$$D_x \left(\frac{1}{k} D_x \right) (kE_x) + D_z^2 E_x + kE_x = 0 \qquad \text{(b)} \quad (9)$$

$$(D_z^2 + k) \left\{ k \left[D_x^2 + D_z^2 + k \right] E_z \right\} +$$

$$+ \left\{ (D_z^2 D_x E_z)\, (D_x k) \right\} = 0 \qquad \text{(c)}$$

B. Circular Cylindrical Coordinates (r, θ, z). Scale Factors: $h_r = h_z = 1$, $h_\theta = r$.

The quantity k is assumed to be a function of r only. The field components E_r, E_θ, E_z are assumed to be independent of θ. They satisfy the following equations:

$$\left(D_r \frac{1}{r} D_r + \frac{1}{r} D_z^2 + \frac{k}{r}\right) (rE_\theta) = 0 \qquad \text{(a)}$$

$$D_r \left(\frac{1}{k} D_r\right) (rkE_r) - \frac{1}{r} \left[\frac{1}{k} D_r (rkE_r)\right] + r (D_z^2 + k)E_r = 0 \quad \text{(b)}$$

$$(D_z^2 + k) \left\{k \left[D_r r D_r + rD_z^2 + k\right]\right\} E_z + \qquad \text{(10)}$$

$$+ r(D_z^2 D_r E_z) (D_r k) = 0 \qquad \text{(c)}$$

C. Spherical Coordinates (r, ϕ, θ). Scale Factors: $h_r = 1$, $h_\phi = r \sin\theta$, $h_\theta = r$.

The quantity k is assumed to be a function of r only. The field components E_r, E_ϕ, E_θ are assumed to be independent of ϕ. They satisfy the following equations:

$$\left\{\sin\theta\, D_r^2 + D_\theta \left(\frac{1}{r^2 \sin\theta} D_\theta\right) + k \sin\theta \right\} (r \sin\theta E_\phi) = 0 \quad \text{(a)}$$

$$\left[\sin\theta\, D_r \left(\frac{1}{k} D_r \right) (r^2 k E_r)\right] + \left[D_\theta \frac{1}{\sin\theta} D_\theta + r^2 k \sin\theta\right] E_r = 0 \quad \text{(b)}$$

(11)

$$D_\theta \sin\theta\, D_r E_\theta = (D_\theta \sin\theta D_\theta + k \sin\theta)\, E_r \qquad \text{(c)}$$

The latter equation is from Equation III(i_1) of Appendix III.

The sets of Equations (9), (10), and (11) are suitable for application in a variety of interior and exterior boundary value problems, e.g., waveguide and scattering problems. The limitation that the field components must be independent of u_2, however, restricts their use somewhat. As was mentioned in Appendix III, analogous sets of equations can be derived without the aforementioned restriction, but the equations can become quite cumbersome. Only the set (9) will receive further attention in this report.

V. THE REFLECTION AND TRANSMISSION COEFFICIENTS FOR PLANE WAVES.

In this section formulas will be given for the reflection and transmission coefficients for a plane wave obliquely incident on the plane interface separating a homogeneous medium from a stratified medium. The formulas will be derived in Appendix V on the basis of the following assumptions:

(i) The medium is stratified in the x-direction; i.e., k is a function of x alone.

(ii) The stratified medium is bounded by infinite parallel planes at $x = 0$ and at $x = d$. The plane wave is assumed to be incident on the $x = 0$ plane, from the negative x-direction.

(iii) The plane of incidence of the incident plane wave is the x-z plane so that all field components are independent of the y-coordinate and can therefore be determined from the system (9). (There is no loss of generality in this assumption, since the y-axis can always be taken to lie in the $x = 0$ plane and orthogonal to the direction of propagation of the plane wave.)

(iv) The electric and magnetic field components E_y, E_z, H_y, H_z are continuous everywhere. The boundary conditions on these field variables are that they are continuous at $x = 0$, and at $x = d$, for every value of z.

The derivation of the reflection and transmission coefficients will employ the traditional method of considering two separate cases: the case in which the incident electric field is polarized perpendicular to the plane of incidence (TE-wave case)* and the case in which the magnetic

* These designations TE-wave and TM-wave are adopted from Born and Wolf.

field is polarized perpendicular to the plane of incidence (TM-wave case). Since the incident electromagnetic field can always be resolved into components described by these two cases, the reflection and transmission coefficients will be completely determined by the formulas derived in these two cases.

A summary of the formulas derived in Appendix V is given below (refer to Figs. 1 and 2 in Appendix V for the meaning of the symbols).

TE-WAVE CASE

$$D_1 = ik_3 \cos\theta_3 \begin{vmatrix} \phi_+(o) & \phi_-(o) \\ \phi_+(d) & \phi_-(d) \end{vmatrix} - \begin{vmatrix} \phi'_+(o) & \phi'_-(o) \\ \phi'_+(d) & \phi'_-(d) \end{vmatrix}$$

$$D_2 = ik_3 \cos\theta_3 \begin{vmatrix} \phi_+(o) & \phi_-(o) \\ \phi_+(d) & \phi_-(d) \end{vmatrix} - \begin{vmatrix} \phi_+(o) & \phi_-(o) \\ \phi'_+(d) & \phi'_-(d) \end{vmatrix}$$

$$\Delta = D_1 + (ik_1 \cos\theta_1)\, D_2$$

$$B_1 = \frac{-D_1 + (ik_1 \cos\theta_1)}{\Delta} A_1$$

$$A_2 = \frac{2ik_1 \cos\theta_1 \left[\phi'_-(d) - ik_3 \cos\theta_3\, \theta_-(d)\right]}{\Delta} A_1$$

$$B_2 = \frac{2ik_1 \cos\theta_1 \left[\phi'_+(d) - ik_3 \cos\theta_3\, \phi_+(d)\right]}{\Delta} A_1$$

$$A_3 = \frac{-2ik_1 \cos\theta_1 \left[\phi_+(d)\, \phi'_-(d) - \phi'_+(d)\, \phi_-(d) \right]}{\Delta} A_1 e^{-ik_3 d \cos\theta_3}$$

$$R_\perp = \left| \frac{B_1}{A_1} \right|^2$$

$$T_\perp = \left| \frac{A_3}{A_1} \right|^2 \quad \frac{\beta_3 \cos\theta_3}{k_1 \cos\theta_1}$$

TM-WAVE CASE

$$G_1 = \frac{ik(d)}{k_3} \cos\theta_3 \quad \begin{vmatrix} \rho'_+(o) & \rho'_-(o) \\ \rho_+(d) & \rho_-(d) \end{vmatrix} - \begin{vmatrix} \rho'_+(o) & \rho'_-(o) \\ \rho'_+(d) & \rho'_-(d) \end{vmatrix}$$

$$G_2 = \frac{ik(d)}{k_3} \cos\theta_3 \quad \begin{vmatrix} \rho_+(o) & \rho_-(o) \\ \rho_+(d) & \rho_-(d) \end{vmatrix} - \begin{vmatrix} \rho_+(o) & \rho_-(o) \\ \rho'_+(d) & \rho'_-(d) \end{vmatrix}$$

$$\delta = G_1 + \left[\frac{ik(o)}{k_1} \cos\theta_1 \right] G_2$$

$$\beta_1 = \left\{ \frac{-G_1 + \left(\frac{ik(o)}{k_1} \cos\theta_1 \right)}{\delta} \right\} \alpha$$

$$\alpha_2 = \frac{-2 \, \frac{ik(o)}{k_1} \cos \theta_1 \left[\rho'_-(d) - \frac{ik(d)}{k_3} \cos \theta_3 \, \rho_-(d) \right]}{\delta} \, \alpha_1$$

$$\beta_2 = \frac{2 \, \frac{ik(o)}{k_1} \cos \theta_1 \left[\rho'_+(d) - \frac{ik(d)}{k_3} \cos \theta_3 \, \rho_+(d) \right]}{\delta} \, \alpha_1$$

$$\alpha_3 = \frac{-2 \, \frac{ik(o)}{k_1} \cos \theta_1 \left[\rho_+(d) \, \rho'_-(d) - \rho'_+(d) \, \rho_-(d) \right]}{\delta} \, \alpha_1 e^{-ik_3 d \cos \theta_3}$$

$$R_{11} = \left| \frac{\beta_1}{\alpha_1} \right|^2$$

$$T_{11} = \left| \frac{\alpha_3}{\alpha_1} \right|^2 \frac{\cos \theta_3}{\cos \theta_1}$$

An important difference between the two cases is the following:

> The TE-wave case depends on the form of the functions $\phi_+(x)$ and $\phi_-(x)$ which are solutions to the differential equation $\phi'' + (k - \gamma^2) \phi = 0$, while the TM-wave case depends on the form of the functions $\rho_+(x)$ and $\rho_-(x)$ which are solutions to the differential equation $(\frac{1}{k} \rho')' + (k - \xi^2) \frac{\rho}{k} = 0$ (cf. equations V(a), V(p), V (q)). Note that the two differential equations are the same when k is constant. Hence, this distinction between the two cases is introduced when we allow k to vary in space.

The case in which medium II is homogeneous, with propagation constant k_2, is so instructive for its lessons in the inhomogeneous case that we introduce it here. For this case, the following relations hold:

$$\phi_{\pm}(x) = e^{\pm ik_2 x \cos \theta_2} = \rho_{\pm}(x); \; k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_3 \sin \theta_3.$$

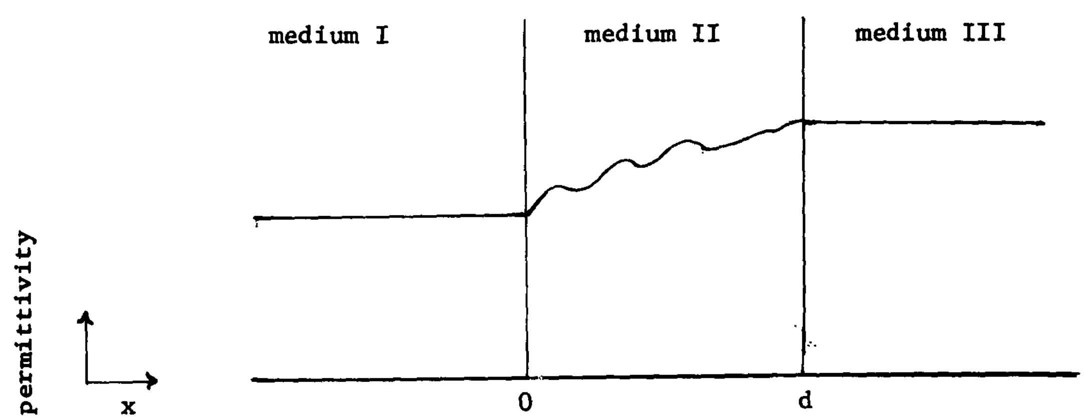
TE-WAVE CASE

$$D_1 = 2k_2 \cos\theta_2 \left[-k_3 \cos\theta_3 \cos(k_2 d \cos\theta_2) + ik_2 \cos\theta_2 \sin(k_2 d \cos\theta_2) \right]$$

$$D_2 = 2i \left[ik_3 \cos\theta_3 \sin(k_2 d \cos\theta_2) + k_2 \cos\theta_2 \cos(k_2 d \cos\theta_2) \right]$$

$$B_2 = \frac{2k_1 \cos\theta_1 \left[k_3 \cos\theta_3 - k_2 \cos\theta_2 \right]}{\Delta}$$

Now note that B_2, for example, is the amplitude of the wave propagating backward in medium II, essentially the reflected wave from the interface between medium II and medium III. This reflection can be extinguished by having $k_3 = k_2$, whence $k_3 \cos\theta_3 = k_2 \cos\theta_2$ (consequence of Snell's Law) and $B_2 = 0$. The latter observation suggests a means for our determining the behavior of the disturbance in medium II when the latter is inhomogeneous and semi-infinite ($d \rightarrow \infty$). We proceed as follows: let medium III be homogeneous with the propagation constant $\sqrt{k(d) - \gamma^2}$ which medium II would have were it suddenly to become homogeneous at $x = d$, keeping the value of the dielectric permittivity which it attains at $x = d$, as diagrammed below



Then, instead of characterizing the obliquely propagating wave by a propagation constant $k_3 \cos \theta_3$ in medium III, we replace it by a quantity $\sigma(d)$, dependent only on d. It can be seen then that B_2 (reflection amplitude in medium II)

$$B_2 = \frac{2ik_1 \cos \theta_1 \left[\phi'_+(d) - i\,\sigma(d)\,\phi_+(d)\right]}{\Delta} A_1.$$

Now, suppose that the ordinary differential equation determining $\phi(x)$ admits of two solutions $\phi_+(x)$ and $\phi_-(x)$ with the property that

$$\lim_{d \to \infty} \left[\phi'_{\pm}(d)\right] = \pm\, \sigma(d)\, \phi_{\pm}(d).$$

That is, for large values of d, we may replace $\phi'_{\pm}(d)$ by $\pm\, i\, \sigma(d)\, \phi_{\pm}(d)$, to a good approximation. Then $B_2 \to 0$ as $d \to \infty$. Let us now calculate the reflection coefficient in medium I, for large d, using the aforementioned assumption. According to the table on page 13

$$B_1 = \frac{-D_1 + ik_1 \cos \theta_1 D_2}{D_1 + ik_1 \cos \theta_1 D_2} A_1 = \frac{-\phi'_+(o) + ik_1 \cos \theta_1\, \phi_+(o)}{\phi'_+(o) + ik_1 \cos \theta_1\, \phi_+(o)} A_1.$$

It follows from the latter relation that $R_\perp = \left|\frac{B_1}{A_1}\right|^2$ will be equal to unity (total reflection case) if $\phi'_+(o) = 0$ or if $\phi_+(o) = 0$ or if $\cos \theta_1 = 0$ (grazing incidence). When medium II is a homogeneous medium so that $\phi_+(x) = e^{ik_2 x \cos \theta_2}$, the non-trivial total reflection condition becomes $\cos \theta_2 = 0$, in agreement with the familiar result. Note that, from the continuity conditions, Snell's Law yields $k_1 \sin \theta_1 = \gamma = \sqrt{k(o) - \sigma(o)}$. The condition $\phi'_+(o) = 0$ is a kind of eigenvalue condition which will determine $\sigma(o)$, which will in turn determine the angle θ_1 for total reflection. Note, also, that the condition $\phi_+'(o) = 0$ <u>might</u> relate the reflection to the region in the dielectric where the index of refraction vanishes, but only when $\phi_+(x)$ is approximately representable in complex exponential form (propagating wave) near x = 0. Otherwise, almost any condition, unrelated to the index of refraction, might arise. All of these observations apply only to the lossless case, of course.

The above remarks have all applied to the TE-wave case. In the TM-wave case, it is possible to deduce for a semi-infinite medium a condition on the angle of incidence θ_1 such that $R_{11} = 0$. The specific angle of incidence in this case is called the "Brewster angle." In this case we must argue, not on the functional behavior of $\phi(x)$, but on the behavior of a function $\rho(x)$ which is, in general, entirely different from $\phi(x)$. The argument here, however, parallels the preceding one and yields the condition

$$\frac{k(o)}{k_1} \cos \theta_1 = -i \frac{\rho_+'(o)}{\rho_+(o)}$$

for the Brewster angle. When $\rho_+(x) = e^{ik_2 x \cos \theta_2}$, the latter condition becomes $k_2 \cos \theta_1 = k_1 \cos \theta_2$. Combining this with Snell's Law, $k_1 \sin \theta_1 = k_2 \sin \theta_2$, we get $\sin 2\theta_1 = \sin 2\theta_2$. In the non-trivial case, the latter condition implies $\theta_2 = \frac{\pi}{2} - \theta_1$. Substituting this in Snell's Law, we get $k_1 \sin \theta_1 = k_2 \cos \theta_1$, or $\tan \theta_1 = k_2/k_1$, yielding Brewster's angle for the case in which medium II is homogeneous and extends to $x = + \infty$.

It is of considerable interest to note that, when medium II is homogeneous and semi-infinite, no zero-reflection situation can arise in the TE-wave case and no total reflection can arise in the TM-wave case. However, when medium II is allowed to be inhomogeneous, both reflection situations can exist in either case. The examples are too complicated to cite here, but some information on the subject can be found in Reference 7, page 19.

VI. CLOSED-FORM SOLUTIONS OF THE WAVE EQUATIONS.

The question will arise as to what functional forms for k(x) will permit the closed-form solution of Equations (9-a) and (9-b) simultaneously. Aside from the trivial case of k(x) constant there is the case of k(x) proportional to e^{Cx} where C is any constant. The solutions in this case will be given in the form of Bessel functions. In general, exponential variations for k(x) seem to show the most promise in this connection. When k(x) is a linear function of x, Equation (9-a) admits closed-form solutions in terms of Airy or Bessel functions, but Equation (9-b) does not seem, in this instance, to admit closed-form solutions in terms of known functions. The whole question discussed in this section is still under investigation.

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APPENDIX I

Dependence of $\vec{J}$ on $\vec{E}$

The equation for the current density which will be used in this report is that derived by Spitzer (ref. 1, p 21, equation 2-12). In our notation ($\vec{J}$ = current density) and in our system of units (MKS), the aforementioned equation of Spitzer is as follows:

$$\left[\frac{m_e}{n_e e^2}\right] \frac{d\vec{J}}{dt} = \vec{E} + \vec{v} \times \vec{B} + \frac{1}{en_e} \nabla p_e - \frac{J}{en_e} \times \vec{B} - \eta \vec{J}. \qquad \text{(Ia)}$$

The following assumptions will be made:

i. $\vec{v} = 0$ since the medium is assumed to be at rest over all.

ii. A new quantity, ω_p, the plasma frequency, defined by

$\omega_p{}^2 = \frac{n_e e^2}{m_e \epsilon_o}$, will be introduced.

iii. The coefficient η, used in Equation (Ia) is essentially a measure of energy dissipation. We introduce in its place the dissipation parameter ν defined by $\nu = \frac{n_e e^2}{m_e} \eta$.

iv. The most drastic assumption is that the term $\frac{1}{en_e} \vec{J} \times \vec{B}$ is negligible with respect to the other quantities appearing in Equation (Ia). This assumption is essentially that the field amplitudes are small (small-signed theory) and that the medium is isotropic.

Equation (Ia) then reduces to

$$\frac{d\vec{J}}{dt} + \nu\vec{J} = \omega_p{}^2 \epsilon_o E + \left(\frac{e}{m_e}\right) \nabla p_e. \qquad \text{(Ib)}$$

At this point it becomes desirable, mostly for the purpose of simplifying the mathematical treatment, to assume that all of the field quantities vary in time at a single frequency ω, so that for any field variable F whatsoever, $\frac{\partial F}{\partial t} = -i\omega F$. It follows from the latter assumption that Equation (Ib) can be expressed as follows:

$$(-i\omega + \nu)J = \omega_p^2 \epsilon_o E + \left(\frac{e}{m_e}\right)\nabla p_e$$

or

$$J = \left[\frac{i\omega_p^2 \epsilon_o}{\omega + i\nu}\right] E + \left[\frac{i\frac{e}{m_e}}{\omega + i\nu}\right] \nabla p_e. \tag{Ic}$$

On the basis of elementary kinetic theory, one concludes that $p_e = nkT_e$* or $p_e = \omega_p^2 (\epsilon_{ok}/e)T$. It follows that ∇p_e may make an important contribution to J whenever $T_e \nabla n_e$ is sufficiently large. The inclusion of ∇p_e in J would, when substituted into Equation (4), yield an inhomogeneous partial differential equation for E, an equation with quite important consequences. However, in any event it would be necessary to solve the homogeneous equation first in order, for example, to obtain the Green's Function for the boundary-value problem in question. Hence, it has been considered desirable here to consider first the consequences of having $\nabla p_e = 0$. The latter assumption, in our context, is tantamount to considering a zero-temperature plasma. The final formula for J which will be used in connection with Equation (4) is

$$J = \left[\frac{i\omega_p^2 \epsilon_o}{\omega + i\nu}\right] E.$$

---

* cf. e.g., A Sommerfeld, Thermodynamics, etc. pp. 8-12.

APPENDIX II

Hamilton's Characteristic Function (the Eikonal of Bruns)

A brief discussion will be given here of the eikonal (characteristic function of Hamilton) defined by H. Bruns.* Equation (8), introduced in the main body of the report, is

$$\nabla \times \nabla \times E = kE, \tag{IIa}$$

where k is a scalar function of the coordinates and E is a vector function of the coordinates. Taking the divergence of both sides of (IIa) yields

$$0 = \nabla \cdot (kE) = E \cdot \nabla k + k \nabla \cdot E \tag{IIb}$$

whence

$$\nabla \cdot E = - E \cdot (\nabla k/k). \tag{IIc}$$

Now, with the help of (IIc), and the relation $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E$, we may rewrite (IIa) as follows:

$$\nabla^2 E + kE = - \nabla \left[E \cdot \frac{\nabla k}{k} \right] . \tag{IId}$$

If k were constant, Equation (IId) would reduce to the familiar vector Helmholtz equation for E, for which we could give the solution $E = Ae^{i(k_1 x + k_2 y + k_3 z)}$ in cartesian coordinates, where $k_1^2 + k_2^2 + k_3^2 = k$, and A is a constant amplitude. With the latter remark in mind, we try to "fit" a solution of the form

$$E = Ae^{iS} \tag{IIe}$$

to the Equation (IId), where A and S are now permitted to vary in space. We substitute (IIe) into (IId) and ascertain the conditions under which (IIe) will be, approximately, a solution to Equation (IId). The solution (IIe) is the "geometrical optical" approximation to the solution to

* For a more detailed discussion of the material introduced here, cf. ref. 6, p. 243, and ref. 7, pp. 207-210.

Equation (IId). In cartesian coordinates, the substitution of (IIe) into (IId) yields the result

$$A\left[k - (\nabla S)^2\right] + \left[\nabla^2 A + \nabla\left(A \cdot \frac{\nabla k}{k}\right)\right] + \\ + i\left[A\nabla^2 S + 2(\nabla A)\cdot(\nabla S) + A(\nabla S)\cdot\frac{\nabla k}{k}\right] = 0 \qquad \text{(IIf)}$$

The latter equation will be satisfied approximately in a non-trivial fashion if, approximately, the following equations are simultaneously satisfied.

$$(\nabla S)^2 = k \qquad (g_1)$$

$$\nabla^2 A + \nabla\left(A \cdot \frac{\nabla k}{k}\right) = 0 \qquad (g_2)\ \text{(IIg)}$$

$$A\nabla^2 S + 2(\nabla A)\cdot(\nabla S) + A(\nabla S)\cdot\frac{\nabla k}{k} = 0. \qquad (g_3)$$

The solution S(x,y,z) to Equation (IIg$_1$) is called the eikonal, and Equation (IIg$_1$) is called the eikonal equation. The function S is important in optics because the surfaces S = constant are the surfaces of constant phase, and they may be thought of as the wave-fronts determining the mode of propagation of the waves. The normals to the surfaces of constant phase determine the directions of the "rays" or paths along which the waves propagate.

Obviously, if k and A are constant in space, the system (IIg) is satisfied exactly if

$$(\nabla S)^2 = k \qquad (h_1)$$

$$\nabla^2 S = 0. \qquad (h_2)\ \text{(IIh)}$$

Now, the substitution of (IIe) into (IIb) yields the result

$$A \cdot \frac{\nabla k}{k} + \nabla \cdot A + A \cdot \nabla S = 0. \tag{IIi}$$

Now, if for <u>any reason</u> (e.g., a constant and orthogonal to ∇k) we have $A \cdot \frac{\nabla k}{k} + \nabla \cdot A = 0$, then $A \cdot \nabla S = 0$, which says that the field direction is transverse to the direction of propagation, since ∇S is in the direction of a ray, while A is in the direction of the field. (The case in which k is a function of x alone and $A = \vec{1}_y y - \vec{1}_z z$ is an interesting non-trivial example.)

Other relations analogous to the above can be derived by combining the various equations in other ways. The assumptions under which they can be derived appear to be far less stringent than the assumptions used, for instance, by Sommerfeld in his discussion of the eikonal.

APPENDIX III

Reduction of Equation (8)

In this section Equation (8) will first be expressed in component form. The resulting system of equations will then be transformed to another system after certain restrictions are imposed on the system.

Let u_1, u_2, u_3 represent the three coordinates of an orthogonal coordinate system in three-dimensional space. The distance ("differential arc length") ds between two points (u_1, u_2, u_3) and ($u_1 + du_1$, $u_2 + du_2$, $u_3 + du_3$) very close together is given by the formula

$$ds^2 = h_1^2\, du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 \qquad \text{(IIIa)}$$

where h_1, h_2, h_3 are functions of u_1, u_2, u_3 and are sometimes called the "scale factors" for the given coordinate system. The following table lists the scale factors for some important coordinate systems (cf. ref. 6, p. 47 ff.):

Coordinate system (u_1, u_2, u_3)	h_1	h_2	h_3
Cartesian (x,y,z)	1	1	1
Circular cylindrical (r, θ, z)	1	r	1
Spherical (r, ϕ, θ)	1	r sin θ	r
Elliptical cylinder (z, ξ, η)	1	$c\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}$	$c\sqrt{\frac{\xi^2-\eta^2}{\eta^2-1}}$

The unconventional ordering of the latter two coordinate systems has been used here for later reference. In order to express the components of (8) in convenient form we introduce the differential operators D_1, D_2, D_3 defined by

$$D_i \phi = \frac{\partial \phi}{\partial u_i}, \quad i = 1, 2, 3, \tag{IIIb}$$

where ϕ is an arbitrary function of u_1, u_2, u_3. Let E_i be the component of E in the direction of the coordinate u_i for $i = 1, 2, 3$. Then (cf. ref. 6, p. 50) the vector equation (8) can be expressed in component form as follows

$$- \left[D_2 \left(\frac{h_3}{h_1 h_2} D_2 \right) + D_3 \left(\frac{h_2}{h_3 h_1} D_3 \right) + \left(h_3 \quad \frac{h_2}{h_1} k \right) \right] (h_1 E_1) +$$

$$+ D_2 \left(\frac{h_3}{h_1 h_2} D_1 \right) h_2 E_2 + D_3 \left(\frac{h_2}{h_3 h_1} D_1 \right) h_3 E_3 = 0 \tag{c_1}$$

$$D_1 \left(\frac{h_3}{h_1 h_2} D_2 \right) h_1 E_1 - \left[D_1 \left(\frac{h_3}{h_1 h_2} D_1 \right) + D_3 \left(\frac{h_1}{h_2 h_3} D_3 \right) + \left(h_3 \quad \frac{h_1}{h_2} k \right) \right] (h_2 E_2) +$$

$$+ D_3 \left(\frac{h_1}{h_2 h_3} D_2 \right) h_3 E_3 = 0 \tag{c_2}$$

(IIIc)

$$D_1 \left(\frac{h_2}{h_3 h_1} D_3 \right) h_1 E_1 + D_2 \left(\frac{h_1}{h_2 h_3} D_3 \right) h_2 E_2 -$$

$$- \left[D_1 \left(\frac{h_2}{h_3 h_1} D_1 \right) + D_2 \left(\frac{h_1}{h_2 h_3} D_2 \right) + \left(\frac{h_1 h_2}{h_3} k \right) \right] h_3 E_3 = 0. \tag{c_3}$$

From Equation (8) it also follows that div(kE) = 0, which can be written

$$D_1(h_2 h_3 k E_1) + D_2(h_3 h_1 k E_2) + D_3(h_1 h_2 k E_3) = 0. \tag{IIId}$$

In the light of the remarks made in Section IIIA, no attempt will be made here to reduce the system (IIIc) given above; but we shall consider a special case of frequenc occurrence and of great importantance, namely, the case in which $h_1 = 1$, k a function of u_1 only, and the ratio of h_2 to h_3

is a product of functions, each of which is a function of only one of the coordinates. The latter assumption is satisfied in any generalized cylindrical coordinate system and in spherical coordinates. The following useful relation can then be established:

$$D_i\left(\frac{h_2}{h_3}D_j\right)(h_3\phi) = \frac{h_2}{h_3}D_j\left(\frac{h_3}{h_2}D_i\right)(h_2\phi) \text{ for } i, j = 1,2,3,\ i \neq j \tag{IIIe$_1$}$$

where ϕ is an arbitrary function of the coordinates. The proof of the latter relation proceeds as follows:

$$\begin{aligned}
D_i\left(\frac{h_2}{h_3}D_j\right)(h_3\phi) &= D_i\left(\frac{h_2}{h_3}D_j\right)\frac{h_3}{h_2}h_2\phi) = D_i\left[D_j(h_2\phi) + \frac{h_2}{h_3}(D_j\ \frac{h_3}{h_2})(h_2\phi)\right] \\
&= D_jD_i(h_2\phi) + \frac{h_2}{h_3}(D_j\ \frac{h_3}{h_2})\ D_i(h_2\phi) + D_i\left[\frac{h_2}{h_3}(D_j\ \frac{h_3}{h_2})\right](h_2\phi) \\
&= \frac{h_2}{h_3}D_j\left(\frac{h_3}{h_2}D_i\right)(h_2\phi) + (h_2\phi)\left[D_iD_j\left(\ln\ \frac{h_3}{h_2}\right)\right].
\end{aligned}$$

By virtue of the assumption about the ratio h_3/h_2, the bracketed quantity in the second term in the last line must vanish, yielding the desired result. One may interchange the subscripts 2 and 3 in (IIIe$_1$) to obtain another relation, namely,

$$D_j\left(\frac{h_3}{h_2}D_j\right)(h_2\phi) = \frac{h_3}{h_2}D_j\left(\frac{h_2}{h_3}D_i\right)(h_3\phi) \text{ for } i,j = 1,2,3,\ i \neq j. \tag{IIIe$_2$}$$

By suitable manipulations using (IIIe$_1$) and (IIIe$_2$), one may prove the following:

$$\frac{h_3}{h_2}D_1\left(\frac{h_2}{h_3}D_3\right)(h_2E_3) - D_3\left(\frac{h_2}{h_3}D_1\right)(h_3E_3) = 2(d_1\ln\frac{h_2}{h_3})\ D_3(h_2E_3) \quad (f_1) \tag{IIIf}$$

$$\frac{h_2}{h_3} D_1\left(\frac{h_3}{h_2} D_2\right)(h_3E_2) - D_2\left(\frac{h_3}{h_2} D_1\right)(h_2E_2) = 2(D_1 \ln \frac{h_3}{h_2}(h_3E_2) \qquad (f_2)$$

(IIIf)

Now, starting with Equation ($IIIc_1$), we obtain

$$\left[D_2\left(\frac{h_3}{h_2} D_2\right) + D_3\left(\frac{h_2}{h_3} D_3\right) + h_3h_2k\right] E_1 - D_3\left(\frac{h_2}{h_3} D_1\right)(h_3E_3) =$$

$$= D_2\left(\frac{h_3}{h_2} D_1\right)(h_2E_2) = \frac{h_3}{h_2} D_1\left(\frac{h_2}{h_3} D_2\right)(h_3E_2).$$

Now Equation (IIId) becomes, with $h_1 = 1$, and k a function of u_1 alone,

$$\frac{1}{k} D_1(h_2h_3kE_1) + D_2(h_3E_2) + D_3(h_2E_3) = 0. \qquad \text{(IIIg)}$$

Solving (IIIg) for $D_2(h_3E_2)$ and substituting the result into the last expression in the last line above, we get

$$\left[D_2\left(\frac{h_3}{h_2} D_2\right) + D_3\left(\frac{h_2}{h_3} D_3\right) + h_3h_2k\right] E_1 - D_3\left(\frac{h_2}{h_3} D_1\right)(h_3E_3) =$$

$$= -\frac{h_3}{h_2} D_1\left(\frac{h_2}{h_3}\frac{1}{k} D_1\right)(h_2h_3kE_1) - \left(\frac{h_3}{h_2} D_1\right)\frac{h_2}{h_3} D_3(h_2E_3).$$

The latter equation, when rearranged, yields, with the help of ($IIIf_1$),

$$\frac{h_3}{h_2} D_1\left(\frac{h_2}{h_3}\frac{1}{k} D_1\right)(h_2h_3kE_1) + \left[D_2\left(\frac{h_3}{h_2} D_2\right) + D_3\left(\frac{h_2}{h_3} D_3\right) + h_3h_2k\right] E_1 =$$

$$= D_3\left(\frac{h_2}{h_3} D_1\right)(h_3E_3) - \frac{h_3}{h_2} D_1\left(\frac{h_2}{h_3} D_3\right)(h_2E_3) = 2(D_1 \ln \frac{h_3}{h_2}) D_3(h_2E_3).$$

($IIIh_1$)

Similarly, starting in a different way from (IIIc$_1$),we may prove

$$\frac{h_2}{h_3} D_1 \left(\frac{h_3}{h_2} \frac{1}{k} D_1 \right) (h_2 h_3 k E_1) + \left[D_2 \left(\frac{h_3}{h_2} D_2 \right) + D_3 \left(\frac{h_2}{h_3} D_3 \right) + h_3 h_2 k \right] E_1 =$$

$$= 2 \left(D_1 \ell n \frac{h_2}{h_3} \right) D_2 (h_3 E_2). \qquad \text{(IIIh}_2\text{)}$$

Now, in cartesian and spherical coordinates, Equation (IIIh$_1$) or (IIIh$_2$) already yields an equation for E_1, since, in these cases $D_1 \left(\ell n \frac{h_2}{h_3} \right) = 0$. In circular cylindrical coordinates, however, a complication still remains, since $- D_1 \ell n \frac{h_3}{h_2} = D_r \ell n\, r = \frac{1}{r} \neq 0$. In order to avoid even more tortuous manipulations, we shall at this point make the additional assumption that all variables are independent of u_2. This is tantamount to setting $D_2 = 0$ wherever D_2 appears in the equations. Equation (IIIh$_2$) then reduces to

$$\frac{h_2}{h_3} D_1 \left(\frac{h_3}{h_2} \frac{1}{k} D_1 \right) (h_2 h_3 k E_1) + \left(D_3 \left(\frac{h_2}{h_3} D_3 \right) + h_3 h_2 k \right) E_1 = 0. \qquad \text{(IIIj)}$$

Also, we obtain immediately from Equation (IIIc$_2$) the equation

$$\left[D_1 \left(\frac{h_3}{h_2} D_1 \right) + D_3 \left(\frac{1}{h_2 h_3} D_3 \right) + \frac{h_3}{h_2} \right] (h_2 E_2) = 0, \qquad \text{(IIIk)}$$

an equation for E_2 alone. Equations (IIIc$_1$) and (IIIc$_3$) reduce to

$$\left[D_3 \left(\frac{h_2}{h_3} D_3 \right) + h_3 h_2 k \right] E_1 = D_3 \left(\frac{h_2}{h_3} D_1 \right) (h_3 E_3) \qquad (\ell_1)$$

$$\text{(III}\ell\text{)}$$

$$D_1 \left(\frac{h_2}{h_3} D_3 \right) E_1 = \left[D_1 \left(\frac{h_2}{h_3} D_1 \right) + \frac{h_2}{h_3} k \right] (h_3 E_3). \qquad (\ell_2)$$

By further algebraic manipulations the latter system can be solved to obtain an equation for E_3 alone. From a practical standpoint, however, after E_1 is obtained from Equation (III_j), E_3 is easily obtainable from Equation ($III\ell_1$) by some simple integrations.

We present here the elimination of E_1 from the system ($III\ell$) for cartesian and circular cylindrical coordinates. Then $h_3 = 1$, and h_2 is independent of u_3. The system ($III\ell$) becomes

$$(D_3{}^2 + k)E_1 = D_3D_1E_3 \qquad (m_1)$$

(IIIm)

$$D_1h_2D_3E_1 = (D_1h_2D_1 + h_2k)E_3. \qquad (m_2)$$

Multiplying ($IIIm_1$) on the left by the operator $D_1h_2D_3$, ($IIIm_2$) on the left by the operator $(D_3{}^2 + k)$, subtracting the latter from the former, and simplifying, we obtain

$$\left[D_1h_2D_1 + h_2D_3{}^2 + h_2k \right] E_3 + \frac{(D_1k)}{k} h_2D_3E_1 = 0. \qquad \text{(IIIn)}$$

Multiplying (IIIn) on the left by the operator $(D_3{}^2 + k)$, and using Equation ($IIIm_1$), we obtain finally

$$(D_3{}^2 + k)\left[D_1h_2D_1 + h_2D_3{}^2 + h_2k \right] E_3 + \frac{(D_1k)}{k} h_2D_3{}^2D_1E_3 = 0, \qquad \text{(IIIp)}$$

thus obtaining an equation for E_3 alone.

APPENDIX IV

Special Solutions to the Wave Equation

In this special case in which the electromagnetic wave is propagating in the direction of the gradient of k (x-direction) the electric field E will be a function of x alone and will satisfy the ordinary differential equation

$$\frac{d^2E}{dx^2} + kE = 0. \tag{IVa}$$

When k has the function form given by

$$k(x) = \frac{Ax + B}{(Cx + D)^5} \ ; \ \Delta = AD - BC, \tag{IVb}$$

where A, B, C, and D are constants, then Equation (IVa) has the solution*

$$E = \left\{ D_1 Ai \left[- \frac{1}{\Delta^{2/3}} \frac{Ax + B}{Cx + D} \right] + D_2 Bi \left[- \frac{1}{\Delta^{2/3}} \frac{Ax + B}{Cx + D} \right] \right\} (CX + D), \tag{IVc}$$

where D_1 and D_2 are arbitrary constants and $Ai[\xi]$ and $Bi[\xi]$ are the Airy functions of the argument ξ. Now, in general, it is not possible to identify one of the solutions in (IVc) as a "forward-propagating" wave, since such an identification would require an examination of the behavior of the functions for large arguments. However, in this case such behavior need not be relevant to the problem at hand, since the argument may not ever become large in the region of interest of this problem. In the event that k(x) has the form given by (IVb), we may avoid the issue by discussing

* cf. ref. 2, pp. 44 and 45.

only inhomogeneous plane sheets of finite thickness, bounded by semi-infinite, homogeneous media.

The following example is discussed in Reference 3, p. 229. The example is originally due to Schelkunoff.

Suppose E to be given as a function of x of the form

$$E(x) = \cos \beta x + \epsilon e^{i\beta x}, \tag{IVd}$$

where β and ϵ are constants. When $\epsilon \ll 1$, this function describes an essentially standing wave, since the first term will be dominant. However, this function can be written in the form

$$E(x) = A(x)e^{i\phi(x)} \qquad (e_1)$$

$$A(x) = \sqrt{(1+\epsilon)^2 \cos^2 \beta x + \epsilon^2 \sin^2 \beta x} \qquad (e_2) \text{ (IVe)}$$

$$\phi(x) = \arc\tan \frac{\epsilon \sin \beta x}{(1+\epsilon)\cos \beta x} \, . \qquad (e_3)$$

The latter form describes a "propagating" wave.

APPENDIX V

Derivation of the Reflection and Transmission Coefficients for an Infinite, Stratified Slab

Figure 1 shows three media. Medium I extends from $x = -\infty$ to $x = 0$ and is assumed to be homogeneous with a real propagation constant k_1. Medium III extends from $x = d$ to $x = \infty$ and has a propagation constant. $k_3 = \beta_3 + i\alpha_3$. Medium II extends from $x = 0$ to $x = d$, and the behavior of the electric field in this medium is governed by the system (9). (As a matter of fact, the behavior of the electric field in Media I and III is also governed by (9) with $k = k_1^2$ and $k = k_3^2$, respectively.) The X-Z plane is termed the plane of incidence.

The derivation of the reflection and transmission coefficients will be preceded by a derivation of the representations for the reflected and transmitted waves. Following the traditional approach (cf., ref. 6, pp. 492-494) we shall consider two cases: electric field vector normal to the plane of incidence, and electric field vector in the plane of incidence. The reflection and transmission coefficients themselves will be derived from the amplitudes of the reflected and transmitted waves on the basis of the energy propagation relations for plane waves given in Reference 6, p. 281.

CASE I. Electric Field Polarized Normal to the of Incidence (TE-Wave).

In this case $E_z = E_x = 0$ in all three media. The equation governing the behavior of E_y in Medium II is Equation (9a). As indicated in the figure, the electric field in Medium I consists of the incident wave, having the functional form $A_1 e^{ik_1(x \cos \theta_1 + z \sin \theta_1)}$, and the reflected wave, having the functional form $B_1 e^{ik_1(-x \cos \theta_1 + z \sin \theta_1)}$. The electric field in Medium III consists of an outgoing wave (the transmitted wave) having the functional form $A_3 e^{ik_3 (x \cos \theta_3 + z \sin \theta_3)}$. No attempt will be made at this point actually to resolve the field in Medium II into forward and backward propagating waves. Equation (9a) can be solved by separating variables. If the solution E_y is represented by $E_y = \phi(x) e^{i\gamma z}$, where γ is a constant, and $\phi(x)$ is independent of z, then $\phi(x)$ satisfies the ordinary differential equation

$$\frac{d^2\phi}{dx^2} + (k - \gamma^2)\,\phi = 0. \tag{Va}$$

Let $\phi = A_2\, \phi_+(x) + B_2\, \phi_-(x)$ represent the general solution to (Va), where $\phi_+(x)$ and $\phi_-(x)$ are independent solutions to (Va), the subscripts + and - being used here for possible future identification as forward and backward propagating waves. Then the solution to (9a) in Medium II can be expressed as

$$E_y = \left[A_2\, \phi_+(x) + B_2\, \phi_-(x)\right] e^{i\gamma z}. \tag{Vb}$$

It follows from Equations (1b) and (3) that

$$\nabla \times E = i\omega\mu_o H. \tag{Vc}$$

The latter equation, when expressed in component form in the case we are considering, becomes

$$-\frac{dE_y}{dz} = i\mu_o \omega H_x \qquad \text{(a)}$$

$$0 = H_y \qquad \text{(b)} \quad \text{(Vd)}$$

$$\frac{dE_y}{dx} = i\mu_o \omega H_z. \qquad \text{(c)}$$

Hence, H_z is the only component of the tangential component of the magnetic field, and the continuity of H_z as a function of x implies the continuity of $\frac{dE_y}{dx}$ as a function of x. In summary, the boundary conditions at x = 0 and at x = d are

$$E_y \text{ is a continuous function of x at x = 0.} \qquad (e_1)$$

$$\frac{dE_y}{dx} \text{ is a continuous function of x at x = 0.} \qquad (e_2)$$

$$E_y \text{ is a continuous function of x at x = d.} \qquad (e_3) \text{ (Ve)}$$

$$\frac{dE_y}{dx} \text{ is a continuous function of x at x = d.} \qquad (e_4)$$

The ultimate goal in our procedure is to obtain B_1/A_1 and A_3/A_1, from which we may derive the reflection and transmission coefficients.

Application of conditions (Ve_1) and (Ve_2) leads to the following pair of equations:

$$(A_1 + B_1)e^{ik_1 z \sin \theta_1} = \left[A_2 \phi_+(0) + B_2 \phi_-(0)\right] e^{i\gamma z} \qquad (f_1)$$

(Vf)

$$ik_1 \cos \theta_1 (A_1 - B_1)e^{ik_1 z \sin \theta_1} = \left[A_2 \phi_+'(0) + B_2 \phi_-'(0)\right] e^{i\gamma z} \qquad (f_2)$$

where the primes on $\phi_+(x)$ and $\phi_-(x)$ denote differentiation with respect to x. Application of conditions (V-e_3) and (V-e_4) leads to the pair:

$$\left[A_2 \phi_+(d) + B_2 \phi_-(d)\right] e^{i\gamma z} = A_3 e^{ik_3(z \sin\theta_3 + d \cos \theta_3)} \qquad (g_1)$$

(Vg)

$$\left[A_2 \phi_+'(d) + B_2 \phi_-'(d)\right] e^{i\gamma z} = ik_3 \cos \theta_3 A_3 e^{ik_3(z \sin \theta_3 + d \cos \theta_3)} \qquad (g_2)$$

When we multiply through Equation (V-f_1) by $e^{-i\gamma z}$, we obtain the relation $e^{i(k_1 \sin \theta_1 - \gamma)z}$ = constant. The latter relation must hold for all values of z, since (V-f_1) is to hold for all values of z. This situation can prevail only if $k_1 \sin \theta_1 - \gamma = 0$. Similar reasoning on (V-g_1) leads to the conclusion that $k_3 \sin \theta_3 = \gamma$. In summary,

$$k_1 \sin \theta_1 = \gamma = k_3 \sin \theta_3. \qquad \text{(Vh)}$$

The latter can be regarded as a generalization of Snell's Law. Also, in view of the relation (V-h), the exponential factors can be cancelled from (V-f) and (V-g), yielding, after some rearrangement, the following set of equations:

$$\begin{array}{llll}
A_2 \phi_+(0) + B_2 \phi_-(0) & -B_1 & & = A_1 \\
A_2 \phi_+'(0) + B_2 \phi_-'(0) + ik_1 \cos \theta_1 B_1 & & & = ik_1 \cos\theta_1 A_1 \\
A_2 \phi_+(d) + B_2 \phi_-(d) & & -A_3 e^{ik_3 d\cos\theta_3} & = 0 \\
A_2 \phi_+'(d) + B_2 \phi_-'(d) & & -ik_3 \cos \theta_3 A_3 e^{ik_3 d\cos\theta_3} & = 0
\end{array} \qquad \text{(Vi)}$$

The latter set of equations can be regarded as a set of simultaneous linear algebraic equations in the unknowns A_2, B_2, B_1, $A_3 e^{ik_3 d \cos \theta_3}$. The determinant of the unknowns is

$$\Delta = \begin{vmatrix} \phi_+(0) & \phi_-(0) & +1 & 0 \\ \phi_+'(0) & \phi_-'(0) & -ik_1\cos\theta_1 & 0 \\ \phi_+(d) & \phi_-(d) & 0 & +1 \\ \phi_+'(d) & \phi_-'(d) & 0 & +ik_3\cos\theta_3 \end{vmatrix} \qquad \text{(Vj)}$$

$$\begin{vmatrix} \phi_+'(0) & \phi_-'(0) \\ \phi_+'(d) & \phi_-'(d) \end{vmatrix} + ik_3\cos\theta_3 \begin{vmatrix} \phi_+'(0) & \phi_-'(0) \\ \phi_+(d) & \phi_-(d) \end{vmatrix} - ik_1\cos\theta_1 \begin{vmatrix} \phi_+(0) & \phi_-(0) \\ \phi_+'(d) & \phi_-'(d) \end{vmatrix}$$

$$-k_1k_3\cos\,\theta_1\cos\,\theta_3 \begin{vmatrix} \phi_+(0) & \phi_-(0) \\ \phi_+(d) & \phi_-(d) \end{vmatrix}$$

Let the minor of the element in the first row and third column of Δ be denoted by D_1, and let the minor of the elements in the second row and third column of Δ be denoted by D_2. Then

$$D_1 = \begin{vmatrix} \phi_+'(0) & \phi_-'(0) & 0 \\ \phi_+(d) & \phi_-(d) & 1 \\ \phi_+'(d) & \phi_-'(d) & ik_3\cos\,\theta_3 \end{vmatrix} = ik_3\cos\theta_3 \begin{vmatrix} \phi_+'(0) & \phi_-'(0) \\ \phi_+(d) & \phi_-(d) \end{vmatrix} - \begin{vmatrix} \phi_+'(0) & \phi_-'(0) \\ \phi_+'(d) & \phi_-'(d) \end{vmatrix} \qquad (k_1)$$

(Vk)

$$D_2 = \begin{vmatrix} \phi_+(0) & \phi_-(0) & 0 \\ \phi_+(d) & \phi_-(d) & 1 \\ \phi_+'(d) & \phi_-'(d) & ik_3\cos\theta_3 \end{vmatrix} = ik_3\cos\theta_3 \begin{vmatrix} \phi_+(0) & \phi_-(0) \\ \phi_+(d) & \phi_-(d) \end{vmatrix} - \begin{vmatrix} \phi_+(0) & \phi_-(0) \\ \phi_+'(d) & \phi_-'(d) \end{vmatrix} \qquad (k_2)$$

Then, (Vd) $\Delta = D_1 + (ik_1\cos\theta_1)D_2$.

Now, solving (Vd), by the use of Cramer's Rule, we find that

$$B_1 = \frac{-D_1 + (ik\cos\theta_1)D_2}{D_1 + (ik\cos\theta_1)D_2} A_1 \qquad V(m_1)$$

$$A_2 = \frac{2ik_1\cos\theta_1 \left[\phi_-'(d) - ik_3 \cos\theta_3\ \phi_-(d)\right]}{\Delta} A_1 \qquad V(m_2)$$

$$B_2 = \frac{2ik_1\cos\theta_1 \left[\phi_+'(d) - ik_3\cos\theta_3\ \phi_+(d)\right]}{\Delta} A_1 \qquad V(m_3)$$

$$A_3 e^{ik_3 d\cos\theta_3} = \frac{\begin{vmatrix} \phi_+(0) & \phi_-(0) & -1 & 1 \\ \phi_+'(0) & \phi_-'(0) & ik_1\cos\theta_1 & ik_1\cos\theta_1 \\ \phi_+(d) & \phi_-(d) & 0 & 0 \\ \phi_+'(d) & \phi_-'(d) & 0 & 0 \end{vmatrix}}{\Delta} A_1$$

$$A_3 = -\frac{2ik_1\cos\theta_1 \left[\phi_+(d)\ \phi_-'(d) - \phi_+'(d)\ \phi_-(d)\right]}{\Delta} A_1 e^{-ik_3 d\cos\theta_3} \qquad V(m_4)$$

The expression $\left[\phi_+(x)\ \ \phi_-'(x) - \phi_+'(x)\ \ \phi_-(x)\right]$ is the Wronskian of $\phi_+(x)$ and $\phi_-(x)$ and is therefore, a constant, by Abel's Formula.*

CASE II. Electric Field in the Plane of Incidence. (TM-Wave).

In this case, the magnetic field is normal to the plane of incidence, $H_x = H_z = 0$, and H_y is a continuous function of x everywhere. Equation 1(a), combined with (5) and (7), yields

$$\nabla \times H = \frac{-i}{\omega\mu_o} kE$$

* cf. Hildebrand, Advanced Calculus for Engineers, Prentice-Hall, 1949, p. 31.

which, when expressed in component form, yields the following:

$$\frac{\partial H_y}{\partial z} = \frac{i}{\omega\mu_o} kE_x \qquad (n_1)$$

V(n)

$$\frac{\partial H_y}{\partial x} = \frac{-i}{\omega\mu_o} kE_z \qquad (n_2)$$

Since E_z is a continuous function of x for all values of x and z, it follows from $V(n_2)$ that $\frac{1}{k}\frac{\partial H_y}{\partial x}$ is likewise. In summary, we may say that the boundary conditions to be imposed on H_y are the following:

H_y and $\frac{1}{k}\frac{\partial H_y}{\partial x}$ are continuous functions of x at x = 0 and at x = d. In addition, we note that, since E_x is a solution to the equation 9(b), an equation whose variables are separable, we may express E_x in the form

$$E_x = \psi(x) \; e^{i\xi z} \qquad V(0)$$

where ξ is a constant, and $\psi(x)$ is a function of x alone satisfying the differential equation

$$\frac{d}{dx}\,\frac{1}{k}\,\frac{d}{dx}\,(k\psi) + (k - \xi^2)\psi = 0. \qquad V(p)$$

Combining equations $V(n_1)$ and V(0), we get

$$\frac{\partial H_y}{\partial z} = \frac{i}{\omega\mu_o}\, k\,\psi(x)\; e^{i\xi z}$$

which, upon integration with respect to z, yields

$$H_y = \frac{1}{\xi\omega\mu_o}\, k\,\psi(x)\; e^{i\xi z} = \rho(x)\; e^{i\xi z},$$

where we have introduced $\rho(x)$ for $\frac{1}{\xi\omega\mu_o}\, k\,\psi(x)$.

The procedure in this case then exactly parallels the procedure used in Case I, except that we are now talking about magnetic fields instead of electric fields. In place of the symbols A_1, A_2, A_3, B_1, B_2, $\phi_+(x)$, $\phi_-(x)$, Δ, D_1, D_2, use the symbols α_1, α_2, α_3, β_1, β_2, $\rho_+(s)$, $\rho_-(x)$, δ, G_1, G_2, respectively, where the α's and β's are now magnetic field amplitudes. Because of the slight difference in the form of the boundary conditions, it turns out that a further modification arises in that, in all of the determinants in question (but not in exponential expressions) k_1 should be replaced by $\frac{k(0)}{k_1}$ and k_3 should be replaced by $\frac{k(d)}{k_3}$. Thus, we obtain, for example, comparing $V(m_4)$ and $V(j)$,

$$\alpha_3 = - \frac{2i\,\frac{k(0)}{k_1}\,\cos\theta_1 \left[\rho_+(d)\,\rho_-'(d) - \rho_+'(d)\,\rho_-(d)\right]}{\delta}\,\alpha_1 e^{-ik_3 d\cos\theta_3} \qquad V(r)$$

$$\delta = \begin{vmatrix} \rho_+(0) & \rho_-(0) & 1 & 0 \\ \rho_+'(0) & \rho_-'(0) & \frac{-ik(0)\cos\theta_1}{k_1} & 0 \\ \rho_+(d) & \rho_-(d) & 0 & 1 \\ \rho_+'(d) & \rho_-'(d) & 0 & \frac{ik(d)\cos\theta_3}{k_3} \end{vmatrix} \qquad V(s)$$

Reflection and transmission coefficients. Following the methods used in Ref. 6, p. 496, we have the reflection and transmission coefficients $R_\perp$, $T_\perp$, R_{11}, T_{11} for cases I and II given, respectively, by V(t) and V(u) as follows (assuming k_1 real):

$$R_{\perp} = \left|\frac{B_1}{A_1}\right|^2 = \left|\frac{-D_1 + (ik_1\cos\theta_1)D_2}{D_1 + (ik_1\cos\theta_1)D_2}\right|^2 \qquad (t_1)$$

V(t)

$$T_{\perp} = \left|\frac{A_3}{A_1}\right|^2 = \left|\frac{\mathrm{Re}(k_3\cos\theta_3)}{k_1 \cos \theta_1}\right| = \qquad (t_2)$$

$$= \left(\frac{4k_1\cos\theta_1\mathrm{Re}(k_3\cos\theta_3)}{|\Delta_e k_3 d \cos\theta_3|^2}\right)\left|\phi_+(d)\,\phi_-'(d) - \phi_+'(d)\,\phi_-(d)\right|^2$$

$$R_{11} = \left|\frac{\beta_1}{\alpha_p}\right|^2 = \left|\frac{-G_1 + (ik(0)/k_1 \cos \theta_1)G_2}{G_1 + (ik(0)/k_1 \cos \theta_1)G_2}\right|^2 \qquad (u_1)$$

V(u)

$$T_{11} = \frac{\alpha_3}{\alpha_1}^2 \quad \frac{\cos\theta_3}{\cos\theta_1} =$$

$$= \frac{\left(\frac{4k^2(0)}{k_1^2}\left|\rho_+(d)\,\rho_-'(d) - \rho_+'(d)\,\rho_-(d)\right|^2\right)}{|\delta_e k_3 d \cos\theta_3|^2} \cos\theta_1\cos\theta_3 \qquad (u_2)$$

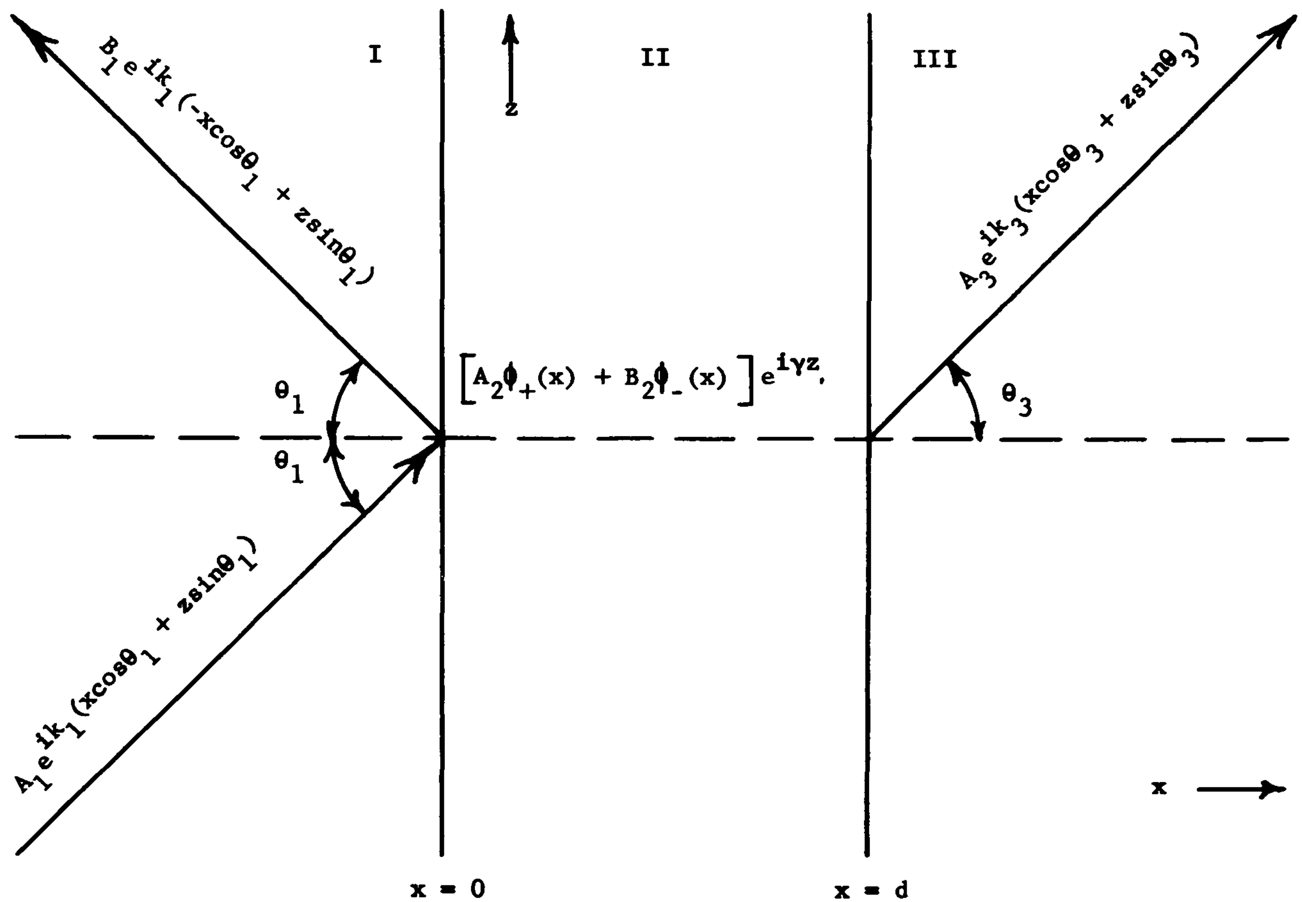

Fig. 1 Transmission and Reflection (TE-wave case)

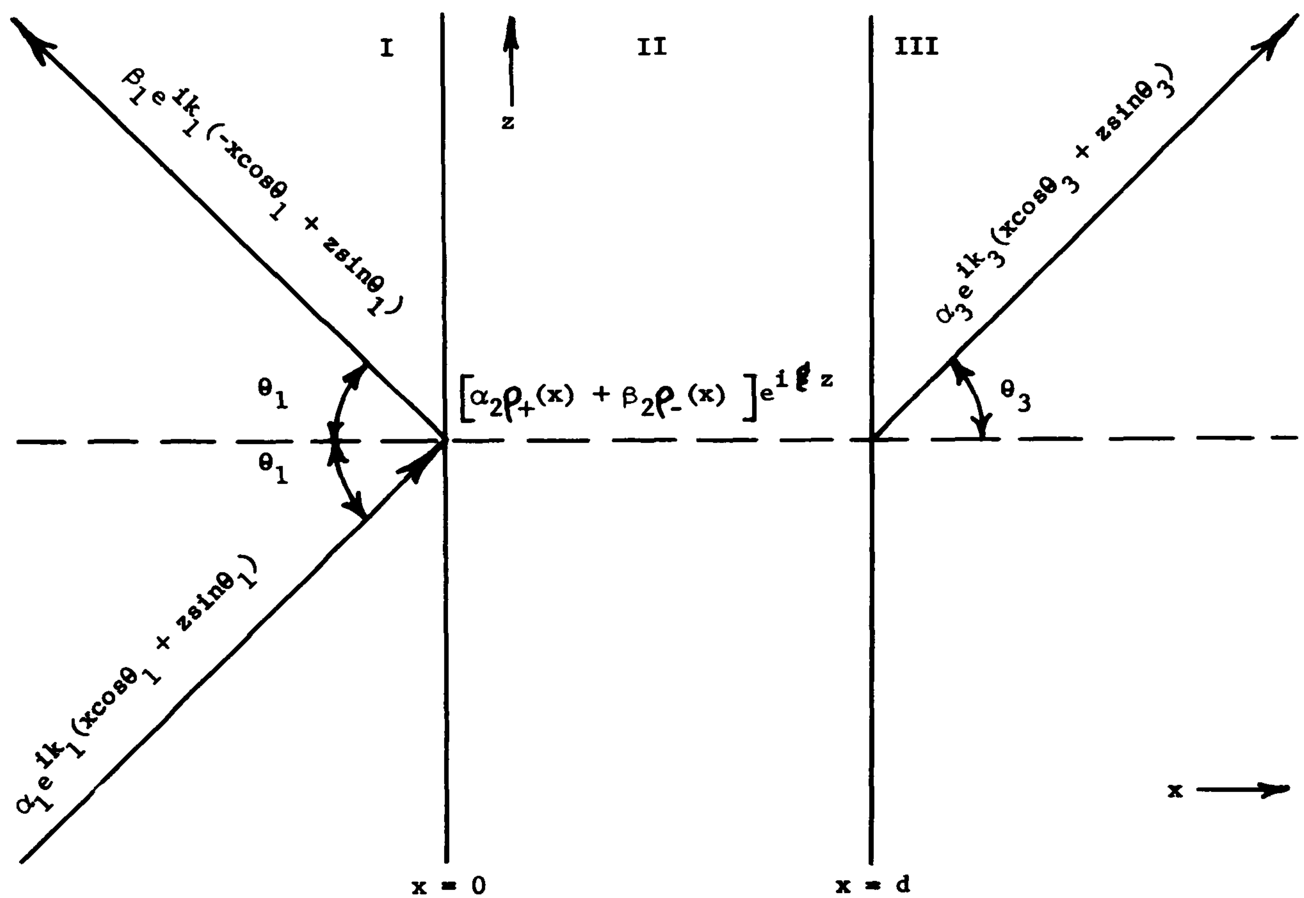

Fig. 2 Transmission and Reflection (TM-wave case)